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REGULAR PARTIALLY INVARIANT SOLUTIONS OF RANK 0 AND DEFECT 1 OF EQUATIONS OF AXISYMMETRIC MOTIONS OF A VISCOUS HEAT-CONDUCTING PERFECT GAS

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All partially invariant solutions of rank 0 and defect 1 of the equations of axisymmetric motions of a viscous heat-conducting perfect gas with a polytropic equation of state that are nonreduced to invariant solutions are described. The gas motions corresponding to these solutions in time and space are presented.

1. Problem Formulation. A system of equations describing axisymmetric motions of a viscous heat-conducting perfect gas with a polytropic equation of state is considered:

$$\rho(u_t + uu_r + wu_z) = -p_r - (2/3)(\mu(u_r + w_z + u/r))_r + 2(\mu u_r)_r + (\mu(u_z + w_r))_z + 2\mu(u/r)_r,$$

$$\rho(w_t + uw_r + ww_z) = -p_z - (2/3)(\mu(u_r + w_z + u/r))_z + 2(\mu w_z)_z + (\mu(u_z + w_r))_r + \mu(u_z + w_r)/r,$$

$$\rho_t + u\rho_r + w\rho_z + \rho u_r + \rho w_z + \rho u/r = 0, \tag{1.1}$$

$$\varepsilon_t + u\varepsilon_r + w\varepsilon_z + p(u_r + w_z + u/r) = (æT_r)_r + (æT_z)_z + æT_r/r$$

$$+ \mu((4/3)(u_r^2 - u_r w_z + w_z^2 + u(u/r - u_r - w_z)/r) + (w_r + u_z)^2).$$

Here u and w are the velocity-vector components written in cylindrical coordinates r, z, and ρ is the density, T is the temperature, $p = R\rho T$ is the pressure, $\mu = m_0 T^\omega$ is the viscosity, $æ = æ_0 T^\omega$ is the thermal conductivity, $\varepsilon = c_V T$ is the internal energy, c_V is the specific heat capacity at constant volume, ω is a constant, R is the universal gas constant, and t is the time.

This paper investigates exact solutions of system (1.1) obtained by methods of group analysis of differential equations [1]. Examples of constructing exact solutions by group methods for equations of a viscous incompressible liquid are described in [2] and for viscous heat-conducting gas equations in [3–5]. The goal of the present work is to construct new partially invariant solutions of rank 0 and defect 1 which are nonreduced to invariant solutions. Some results presented below were formulated in [6].

It was shown in [3] that system (1.1) admits the Lie algebra L_5, and its basis was written. Using formulas for recalculation of basis operators with substitution of variables, we can easily reveal that the basis operators in the variables t, r, z, u, w, ρ, and T have the form

$$X_1 = \partial_z, \quad X_2 = t\partial_z + \partial_w, \quad X_3 = \partial_t, \quad X_4 = t\partial_t + r\partial_r + z\partial_z - \rho\partial_\rho,$$
$$X_5 = r\partial_r + z\partial_z + u\partial_u + w\partial_w + 2(\omega - 1)\rho\partial_\rho + 2T\partial_T. \tag{1.2}$$

To simplify system (1.1), we can assume that $R = 1$ and $m_0 = 1$ with accuracy up to equivalence transformations found in [3]. The optimal system of subalgebras of the Lie-algebra L_5 was constructed in [3]. Let us consider four-dimensional subalgebras.

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TABLE 1

Subalgebra	Subalgebra basis	The subalgebra invariants
4.1	1, 2, 3, $4+\alpha 5$ ($\alpha \neq -1$)	$I_1 = u^{1+\alpha} r^{-\alpha}$, $I_2 = \rho^{1+\alpha} r^{2\alpha(1-\omega)+1}$, $I_3 = T^{1+\alpha} r^{-2\alpha}$
4.1a	1; 2; 3; $4-5$ ($\alpha = -1$)	$I_1 = r$, $I_2 = \rho u^{1-2\omega}$, $I_3 = Tu^{-2}$
4.2	1, 3, 4, 5	$I_1 = w/u$, $I_2 = \rho r u^{1-2\omega}$, $I_3 = Tu^{1-2\omega}$
4.3	1, 2, 4, 5	$I_1 = ut/r$, $I_2 = \rho t^{2\omega-1} r^{2(1-\omega)}$, $I_3 = Tt^2 r^{-2}$
4.4	1, 2, 3, 5	$I_1 = u/r$, $I_2 = \rho r^{2(1-\omega)}$, $I_3 = Tr^{-2}$

The optimal system of four-dimensional subalgebras of the Lie-algebra L_5 is presented in Table 1. The second column shows the numbers of basis operators from the complete basis (1.2) ($4+\alpha 5$ denotes a linear combination of the operators $X_4 + \alpha X_5$). The complete set of functionally independent invariants for each subalgebra is listed in the third column.

Partially invariant solutions for subalgebras 4.1–4.4 have rank 0 and defect 1, except for subalgebra 4.1 whose solutions have rank 1 and defect 2 and are not considered here. In the present work, all partially invariant solutions of rank 0 and defect 1 are constructed, and their properties are analyzed. Substitution of the solution representation for subalgebras into the initial system yields a four-equation system with partial derivatives of one function. The system is overspedetermined, and it is necessary to analyze it for compatibility. The compatibility analysis of overdetermined systems is a rather complex task and requires a large volume of analytical transformations; therefore, all computations were carried out on a personal computer using the Reduce language of analytical computations. During the investigations, we determined the system compatibility conditions, the specific form of the solution, as well as the relations between the constants that enter into the solution representation. The following conditions are imposed on the solutions obtained:

$$\rho > 0, \qquad T > 0, \qquad æ \geqslant 0. \tag{1.3}$$

Solutions not satisfying conditions (1.3) are not considered.

For the solutions obtained, the problem of reduction of partially invariant solutions is studied, which is formulated as follows. Let a partially invariant solution of rank s and defect δ be given. It is necessary to find whether there exists a subgroup for which the solution is also partially invariant; the relationships $s' = s$ and $\delta' < \delta$ are valid for rank s' and defect δ' of this solution. The importance of studying the reduction problem is determined by the fact that, generally speaking, it is simpler to find solutions with a smaller defect. Partially invariant solutions of rank 0 and defect 1 are analyzed in the present work. Since the solution rank should not increase during reduction, it is possible to speak only about reduction to invariant solutions of rank 0 described in detail in [3]. Therefore, if reduction of the solutions obtained is shown, their further investigation is not interesting.

Further, we consequently consider subalgebras shown in Table 1 and analyze the compatibility of the overdetermined systems arising. The problem of reduction of the solutions obtained is investigated, and the possibility of reduction to invariant solutions is shown for some cases. For partially invariant solutions nonreduced to invariant solutions, some properties of the corresponding gas motions are investigated.

2. Subalgebra 4.1. The solution of the problem is presented in the following form:

$$u = u_0 r^{\beta}, \quad \rho = \rho_0 r^{2\omega\beta-\beta-1}, \quad T = T_0 r^{2\beta}, \quad w = w(t,r,z), \quad \beta = \alpha/(1+\alpha). \tag{2.1}$$

Substituting presentation (2.1) into system (1.1), we obtain

$$w_{rz} - 4\omega\beta w_z/r + (4(\beta u_0(2\omega\beta+\beta-1) - u_0) - 3T_0^{-\omega}\rho_0(T_0(2\omega\beta+\beta-1) + \beta u_0^2))r^{\beta-2} = 0;$$

$$w_{rr} + 4w_{zz}/3 - T_0^{-\omega}\rho_0 r^{-\beta-1}(w_t + ww_z) + (2\omega\beta + 1 - T_0^{-\omega}\rho_0 u_0)w_r/r = 0; \tag{2.2}$$

$$w_z + 2\omega\beta u_0 r^{\beta-1} = 0; \tag{2.3}$$

$$w_r^2 + 4w_z^2/3 - (T_0^{1-\omega}\rho_0 + 4u_0(\beta+1)/3)r^{\beta-1}w_z - 2c_V T_0^{1-\omega}\beta u_0 r^{3\beta-2\omega\beta-1}$$

$$+ (4æ_0 T_0\beta^2(\omega+1) - T_0^{1-\omega}\rho_0 u_0(\beta+1) + 4u_0^2(\beta^2-\beta+1)/3)r^{2(\beta-1)} = 0. \tag{2.4}$$

Integrating (2.3), we obtain $w = -2\omega\beta u_0 r^{\beta-1} z + f(t, r)$. Then, (2.4) has the form

$$F(r) + (f_r - 2\omega\beta(\beta-1)u_0 r^{\beta-2} z)^2 = 0. \tag{2.5}$$

Splitting (2.5) in powers of z, we obtain $\omega\beta u_0 = 0$ and $w = g(t) + h(r)$. It should be noted that, at $\omega \neq 0$, only an identically constant function $w \equiv w_0$ is possible. The case $\omega = 0$ is further considered.

Substituting the expression for w into (2.2) and differentiating the latter with respect to t, we have $g'' = 0$, wherefrom $g = c_1 t + c_2$ is obtained. As a result, Eq. (2.2) takes the form

$$rh'' + (1-\lambda)h' - c_1\rho_0 r^{-\beta} = 0, \qquad \lambda = \rho_0 u_0.$$

Solving this equation, we determine the form of the function w:

$$w = \begin{cases} c_3 r^\lambda/\lambda - c_1\rho_0 r^{1-\beta}/[(\beta+\lambda-1)(1-\beta)] + c_1 t + w_0, & \beta \neq 1-\lambda, \\ c_3 r^\lambda/\lambda + c_1\rho_0 r^\lambda(\ln r - 1/\lambda)/\lambda + c_1 t + w_0, & \beta = 1-\lambda. \end{cases} \tag{2.6}$$

Let us substitute (2.6) into (2.4) and split in powers of r. This procedure is sufficiently labor-consuming, but finally we obtain $w \equiv w_0$ for $\beta = 1-\lambda$ and $c_1c_3 = 0$ for $\beta \neq 1-\lambda$.

Let $c_1 = 0$. Then, Eq. (2.4) yields $3\beta - 2\lambda + 1 = 0$, and the solution acquires the final form:

$$w = w_1 r^{(3\beta+1)/2} + w_0, \qquad w_1 = \text{const}, \quad w_0 = \text{const}. \tag{2.7}$$

Let $c_3 = 0$. In this case, splitting of Eq. (2.4) in powers of r (2.4) gives two variants of the solution: $\beta = 1/2$ or $\beta = 1/5$; however, for $\beta = 1/2$, Eq. (2.4) yields $c_V T_0 = 0$, which contradicts conditions (1.3). For $\beta = 1/5$, the solution has the form

$$w = w_1 r^{4/5} + c_1 t + w_0. \tag{2.8}$$

Let us study the possibility of reducing the solutions obtained to invariant ones. Let an arbitrary operator of subalgebra 4.1 considered be written in the form

$$H = a_1X_1 + a_2X_2 + a_3X_3 + a_4(X_4 + \alpha X_5) = (a_3 + a_4 t)\partial_t + a_4(1+\alpha)r\partial_r + (a_1 + a_2 t + a_4(1+\alpha)z)\partial_z$$

$$+ a_4\alpha u\partial_u + (a_2 + a_4\alpha w)\partial_w + a_4(2\alpha(\omega-1)-1)\rho\partial_\rho + 2a_4\alpha T\partial_T, \quad a_i = \text{const } (i = 1, \dots, 4).$$

By virtue of the solution representation (2.1), it suffices to verify the invariance only for the function w. Let us first consider the case of the identically constant function $w \equiv w_0$:

$$H(w - w_0)\Big|_{w\equiv w_0} = a_2 + a_4\alpha w_0 = 0.$$

The equality to zero is reached by choosing $a_2 = -a_4\alpha w_0$. It is easy to find that the solution is invariant with respect to the three-dimensional subalgebra with the basis X_1, X_3, $X_4 + \alpha X_5 - \alpha w_0 X_2$.

Let us investigate the possibility of reduction for the solution of the form $w = w_1 r^{4/5} + c_1 t + w_0$; $\alpha = \beta/(1-\beta) = 1/4$. For this solution, we obtain

$$H(w_1 r^{4/5} + c_1 t + w_0 - w)\Big|_{w=w_1 r^{4/5}+c_1 t+w_0}$$

$$= (a_4 w_1 r^{4/5} + c_1(a_3 + a_4 t) - a_2 - a_4 w/4)\Big|_{w=w_1 r^{4/5}+c_1 t+w_0}$$

$$= 3w_1 a_4 r^{4/5}/4 + 3c_1 a_4 t/4 + c_1 a_3 - a_4 w_0/4 - a_2 = 0.$$

The equality to zero is possible in one of the two cases: 1) $a_2 = -a_4 w_0/4$, $w_1 = 0$, and $c_1 = 0$; 2) $a_4 = 0$ and $a_2 = c_1 a_3$. In the first case, an identically constant solution is obtained; the possibility of its reduction is demonstrated above. For $a_4 = 0$, we obtain invariance with regard to the subalgebra with the basis X_1 $X_2 + c_1 X_3$, but the solution invariant with regard to this subalgebra has rank 1, which does not correspond to the definition of reduction, since the rank should not increase during reduction. Thus, solution (2.8) is not reduced to invariant ones. Similar arguments allow one to conclude that solution (2.7) is not reduced to invariant ones either.

Thus, two families of partially invariant solutions that are not reduced to invariant ones are obtained for subalgebra 4.1.

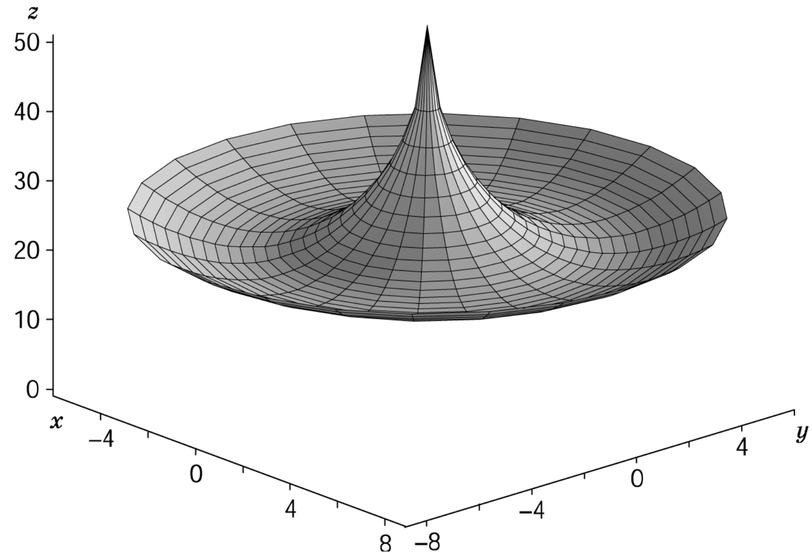

Fig. 1

1. Stationary gas flow with constant viscosity and thermal conductivity ($\omega = 0$). The solution is written in the form

$$u = u_0 r^{\beta}, \quad \rho = \rho_0 r^{-\beta-1}, \quad T = T_0 r^{2\beta}, \quad w = w_1 r^{(3\beta+1)/2} + w_0. \tag{2.9}$$

Solution (2.9) corresponds to a flow from a constant-power source distributed along the Oz axis. For this solution, the values of $\beta \in (0, 1)$ are set, as well as arbitrary values of w_0 and thermodynamic parameters c_V and $æ_0$. The remaining constants entering into the solution representation are expressed through these parameters from the relationships that can be easy obtained from the initial system. The streamlines have the form

$$z = A_1 + 2w_1 r^{(\beta+3)/2}/[(\beta+3)u_0] + w_0 r^{1-\beta}/[(1-\beta)u_0].$$

The streamlines for solution (2.9) are shown in Fig. 1. The whole picture of the flow is obtained by shifting the surface presented in Fig. 1 along the z axis.

2. Nonstationary gas flow with constant viscosity and thermal conductivity ($\omega = 0$). The solution is represented as follows:

$$u = u_0 r^{1/5}, \quad \rho = \rho_0 r^{-6/5}, \quad T = T_0 r^{2/5}, \quad w = w_1 r^{4/5} + c_1 t + w_0. \tag{2.10}$$

Solution (2.10) corresponds to the motion with a constant-power source distributed along the Oz axis. For solution (2.10), the values of $u_0 > 0$, arbitrary values of w_0 and thermodynamic parameters c_V and $æ_0$ are prescribed; the remaining constants entering into the solution representation are expressed through them. The trajectories have the form

$$r(t) = (4u_0(t + A_1)/5)^{5/4}, \quad z(t) = A_2 + 8w_1 t^2/(25\rho_0) + (4w_1 u_0 A_1/5 + w_0)t.$$

Let us consider a sphere filled by a gas at the initial time. In this case, the solution describes sphere deformation into an expanded torus (Fig. 2). Figure 2 shows the positions of domains occupied by the gas at the initial and current times.

3. Subalgebra 4.2. The solution of the problem is represented in the form

$$w = w_0 u, \quad \rho = \rho_0 r^{-1} u^{1-2\omega}, \quad T = T_0 u^2, \quad u = u(t, r, z). \tag{3.1}$$

Substituting representation (3.1) into system (1.1), we obtain

$$4u_{rr} + w_0 u_{rz} + 3u_{zz} + 2\omega(4u_r^2 + w_0 u_r u_z + 3u_z^2)/u + 4(1-\omega)r^{-1}u_r$$

$$- 3\rho_0 T_0^{-\omega}((2\omega T_0 + T_0 + 1)uu_r + w_0 uu_z + u_t)/(ur) + (3\rho_0 T_0^{1-\omega} - 4)r^{-2}u = 0; \tag{3.2}$$

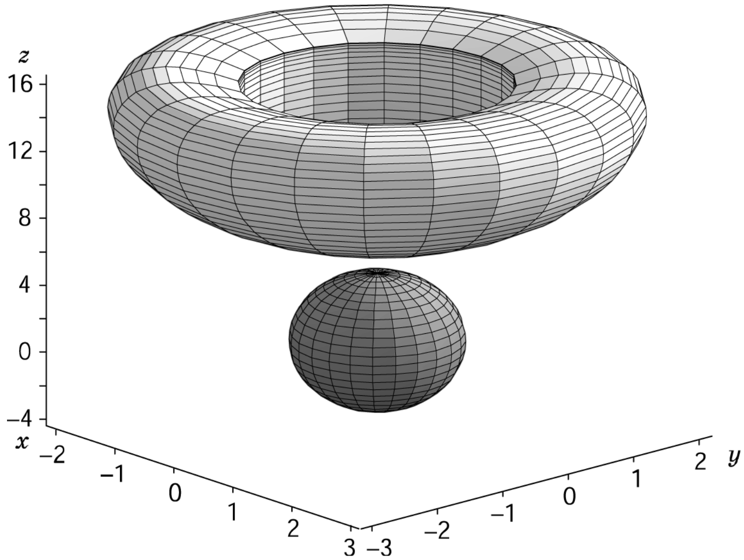

Fig. 2

$$3w_0u_{rr} + u_{rz} + 4w_0u_{zz} + 2\omega(3w_0u_r^2 + u_ru_z + 4w_0u_z^2)/u + (1-4\omega)r^{-1}u_z$$

$$- 3\rho_0T_0^{-\omega}(w_0uu_r + (2\omega T_0 + T_0 + w_0^2)uu_z + w_0u_t)/(ur) + 3w_0r^{-1}u_r = 0; \tag{3.3}$$

$$2\omega u(u_r + w_0u_z) + (2\omega - 1)u_t = 0; \tag{3.4}$$

$$6æ_0T_0(u_{rr} + u_{zz}) + u^{-1}((3w_0^2 + (12\omega + 6)æ_0T_0 + 4)u_r^2 + 2w_0u_ru_z$$

$$+ (4w_0^2 + (12\omega + 6)æ_0T_0 + 3)u_z^2) - 6c_VT_0^{1-\omega}u^{-2\omega}(uu_r + w_0uu_z + u_t)$$

$$+ 6æ_0T_0r^{-1}u_r - (3\rho_0T_0^{1-\omega} + 4)r^{-1}(u_r + w_0u_z) - (3\rho_0T_0^{1-\omega} - 4)r^{-2}u = 0. \tag{3.5}$$

System (3.2)–(3.5) is a system of one first-order equation and three second-order equations with partial derivatives of the function $u(t, r, z)$. Studying compatibility of this system requires more effort than other systems considered in the present paper. To analyze this system, a great number of analytical calculations are necessary: therefore, only the main results are presented below.

3.1. *Nonstationary Case.* Let $\omega \neq 0$. Adding three differential corollaries of Eq. (3.4) to Eqs. (3.2), (3.3), and (3.5), we obtain a system of six second-order quadratic equations linear in major derivatives. All second derivatives u_{tt}, u_{tr}, u_{tz}, u_{rr}, u_{rz}, and u_{zz} can be expressed as quadratic polynomials from the first derivatives u_t, u_r, and u_z with coefficients depending on u. Let the conditions of equality of the third mixed derivatives be written as

$$(u_{tt})_r = (u_{tr})_t, \quad (u_{tt})_z = (u_{tz})_t, \quad (u_{rr})_z = (u_{rz})_r, \quad (u_{rz})_z = (u_{zz})_r,$$

$$(u_{tr})_r = (u_{rr})_t, \quad (u_{tr})_z = (u_{rz})_t, \quad (u_{tz})_r = (u_{rz})_t, \quad (u_{tz})_z = (u_{zz})_t.$$

Substituting the expressions for the second derivatives into these conditions, we obtain eight equations which are third-order polynomials of the first derivatives u_t, u_r, and u_z. These expressions are not presented because of their awkwardness. Combining them, however, we can obtain a corollary that does not include derivatives and is an exponential expression of the function u. An analysis of this corollary reduces to consideration of particular cases $\omega = \pm 1$, $\omega = \pm 2$, $\omega = \pm 1/2$, $\omega = \pm 3/2$, and $\omega = -1/4$. All these cases yield either an identically constant solution or a conflict with conditions (1.3). Thus, there are no nontrivial solutions in the nonstationary case.

3.2. *Stationary Case.* Let $\omega = 0$. Then, Eq. (3.4) takes the form $u_t = 0$, and three second-order equations independent of t remain in system (3.2)–(3.5). In this case, to investigate the system, it is convenient to introduce the following notation:

$$\xi(r, z) = u_r + w_0u_z, \qquad \eta(r, z) = u_r - w_0u_z. \tag{3.6}$$

Then, the system takes the form

$$(3w_0^2+4)u\xi_r + w_0 u\xi_z + (3w_0^2+3)u\eta_z - 3w_0 r^{-1}u\eta$$

$$- (3\rho_0 w_0^2 + 3\rho_0 + 3\rho_0 T_0 - 4)r^{-1}u\xi + (3\rho_0 T_0 - 4)r^{-2}u^2 = 0; \tag{3.7}$$

$$w_0 u\xi_r - u\xi_z + (3\rho_0 T_0 - 1)r^{-1}u\eta + (3\rho_0 T_0 - 4)w_0 r^{-2}u^2 = 0; \tag{3.8}$$

$$6æ_0 T_0 u(\xi_r + \eta_z) + (6æ_0 T_0/(w_0^2+1) + 3)(\xi^2+\eta^2) + \xi^2 - (6æ_0 T_0/(w_0^2+1))w_0 r^{-1}u\eta$$

$$- 6c_V T_0 u^2\xi - (3\rho_0 T_0 - 6æ_0 T_0/(w_0^2+1) + 4)r^{-1}u\xi - (3\rho_0 T_0 - 4)r^{-2}u^2 = 0. \tag{3.9}$$

According to (3.6), the first derivatives of the functions ξ and η are related as $w_0\xi_r - \xi_z + \eta_r + w_0\eta_z = 0$. Considering this fact, from system (3.7)–(3.9), the derivatives ξ_r, ξ_z, η_r, and η_z can be expressed as quadratic polynomials of ξ and η with coefficients depending on u. The conditions of equality of the mixed derivatives $(\xi_r)_z = (\xi_z)_r$ and $(\eta_r)_z = (\eta_z)_r$ are used as compatibility conditions. Substituting the expressions for the derivatives of ξ and η into these conditions, we obtain two equations, which are third-order polynomials of ξ and η. Repeatedly differentiating them with respect to r and z and eliminating the derivatives, we obtain another four corollaries of the system. These expressions are not shown because of their awkwardness. However, there exists their combination in the form $k_1\xi + k_2\eta + k_3 r^{-1}u = 0$, where k_1, k_2, and k_3 are constants. Substituting this expression into system (3.7)–(3.9), we obtain the solution in the form $u = k/r$, where $k = \text{const}$. At the same time, it follows from Eq. (3.7) that $T_0 = -1/2$, which contradicts conditions(1.3), i.e., the solution of the form $u = k/r$ is nonphysical. Thus, there are no nontrivial physical solutions for subalgebra 4.2.

4. Subalgebra 4.3. The problem solution is represented in the form

$$u = u_0 r/t, \quad \rho = \rho_0 r^{2(\omega-1)}t^{1-2\omega}, \quad T = T_0 r^2 t^{-2}, \quad w = w(t,r,z). \tag{4.1}$$

Substituting representation (4.1) into system (1.1), we obtain

$$w_{rz} - (4\omega/r)(w_z - u_0/t) - 3T_0^{-\omega}\rho_0(2\omega T_0 + u_0^2 - u_0)/(rt) = 0;$$

$$w_{rr} + 4w_{zz}/3 - T_0^{-\omega}\rho_0 t(w_t + ww_z)/r^2 + (2\omega + 1 - T_0^{-\omega}\rho_0 u_0)w_r/r = 0; \tag{4.2}$$

$$tw_z + 2\omega(u_0 - 1) + 1 = 0; \tag{4.3}$$

$$w_r^2 + 4w_z^2/3 - (T_0^{1-\omega}\rho_0 + 8u_0/3)w_z/t + 2c_V T_0^{1-\omega}(1-u_0)r^{2-2\omega}t^{2\omega-3}$$

$$+ (4æ_0 T_0(\omega+1) + 4u_0^2/3 - 2T_0^{1-\omega}\rho_0 u_0)t^{-2} = 0. \tag{4.4}$$

Integrating (4.3), we find

$$w = (-2\omega u_0 + 2\omega - 1)z/t + f(t,r). \tag{4.5}$$

Substituting (4.5) into Eq. (4.2) and differentiating it with respect to z, we obtain two variants of the solution: $(u_0 - 1)\omega + 1 = 0$ and $2(u_0 - 1)\omega + 1 = 0$. Let us consider the first variant (similar calculations are used for the second one).

Let us introduce a new variable $s = r/t$ and pass to the function $h(s,t) = f(r,t)$. Then Eqs. (4.2) and (4.4) take the form

$$s^2 h_{ss} + b_2 s h_s - T_0^{-\omega}\rho_0(th_t + h) = 0; \tag{4.6}$$

$$h_s^2 + 2b_0 c_V s^{2(1-\omega)}t + b_1 = 0, \tag{4.7}$$

where b_0, b_1, and b_2 are constants expressed through ρ_0, T_0, ω, and $æ_0$. Investigating Eqs. (4.6) and (4.7) for compatibility, we can obtain a corollary that does not contain the function h and its derivatives. Splitting the corollary in powers of t, the following expression can be derived as compatibility conditions:

$$2T_0^{\omega}(4-\omega^2)\omega - (5\omega - 4)\rho_0 = 0, \qquad b_1 = 0.$$

For $b_1 = 0$, the function h can easily be determined from (4.7). In the initial notation, the sought function w has the form

$$w = z/t + w_1 r^{2-\omega}t^{\omega-3/2}, \quad w_1 = \text{const}. \tag{4.8}$$

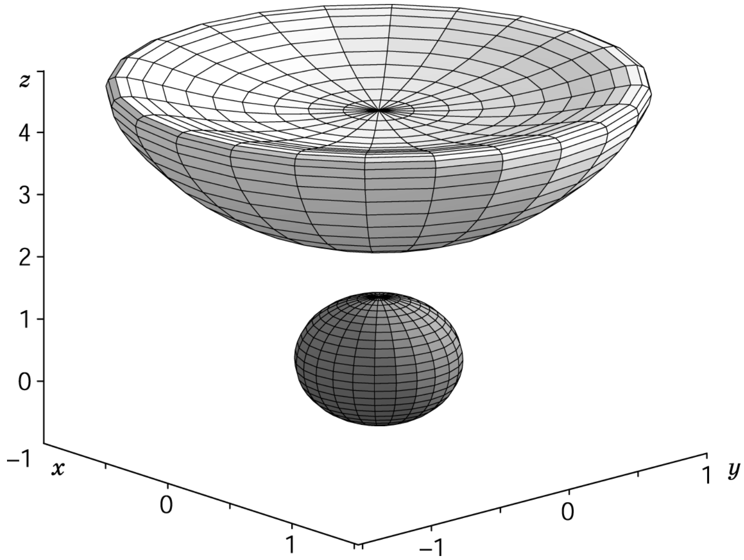

Fig. 3

Similar calculations for the case $2(u_0-1)\omega+1=0$ yield the following solution:

$$w = w_1 r^{2-\omega} t^{\omega-3/2} + w_0, \qquad w_1 = \text{const}, \quad w_0 = \text{const}. \tag{4.9}$$

It is not difficult to verify that solutions (4.8) and (4.9) are not reduced to invariant ones. Let it be shown for (4.9). Let the arbitrary operator of subalgebra 4.3 be written as

$$H = a_1X_1 + a_2X_2 + a_4X_4 + a_5X_5 = a_4 t\partial_t + (a_4+a_5)r\partial_r + (a_1 + a_2 t + (a_4+a_5)z)\partial_z$$

$$+ a_5 u\partial_u + (a_2 + a_5 w)\partial_w + (2(\omega-1)a_5 - a_4)\rho\partial_\rho + 2a_5 T\partial_T, \quad a_i = \text{const} \quad (i=1,\ldots,5).$$

We act on (4.9) by the operator H and verify if the following equality is satisfied:

$$H(w_1 r^{2-\omega} t^{\omega-3/2} + w_0 - w)\Big|_{w=w_1 r^{2-\omega} t^{\omega-3/2}+w_0} = -a_2 - a_5 w_0 + (a_4/2 + a_5(1-\omega))w_1 r^{2-\omega} t^{\omega-3/2} = 0.$$

This equality is valid for $a_2 = -a_5 w_0$ and $a_4 = 2a_5(\omega-1)$. Thus, solution (4.9) is invariant with respect to the two-dimensional subalgebra with the basis X_1, $-w_0X_2 + 2(\omega-1)X_4 + X_5$. The rank of the invariant solution obtained on the basis of this subalgebra is equal to 1; therefore, reduction is impossible in the given case. Similar calculations give the same results for solution (4.8).

Thus, two families of partially invariant solutions nonreduced to invariant solutions are obtained for subalgebra 4.3.

1. Nonstationary gas motion:

$$u = u_0 r/t, \quad \rho = \rho_0 r^{2(\omega-1)} t^{1-2\omega}, \quad T = T_0(r/t)^2, \quad w = z/t + w_1 r^{2-\omega} t^{\omega-3/2}. \tag{4.10}$$

For solution (4.10), the values of $\omega \in (-1,0)$ and c_V are given, and the remaining constants entering into the solution representation are expressed through them. The trajectories have the form

$$r(t) = A_1 t^{(\omega-1)/\omega}, \qquad z(t) = A_2 t + (2\omega/(3\omega-4))w_1 A_1^{2-\omega} t^{(5\omega-4)/(2\omega)}.$$

2. Nonstationary gas motion:

$$u = u_0 r/t, \quad \rho = \rho_0 r^{2(\omega-1)} t^{1-2\omega}, \quad T = T_0(r/t)^2, \quad w = w_1 r^{2-\omega} t^{\omega-3/2} + w_0. \tag{4.11}$$

For solution (4.11), the values of $\omega \in (-1,0)$ and arbitrary values of w_0 and c_V are given; the remaining constants entering into the solution representation are expressed through them. The trajectories have the form

$$r(t) = A_1 t^{(2\omega-1)/(2\omega)}, \qquad z(t) = A_2 + w_0 t + (\omega/(2\omega-1))w_1 A_1^{2-\omega} t^{(2\omega-1)/\omega}.$$

Solution (4.11) makes sense for positive t; therefore, a certain value $t_0 > 0$ is chosen as the initial time. Let us consider a gas-filled sphere at the initial time. In this case, the solution describes deformation of this sphere (Fig. 3).

5. Subalgebra 4.4. The solution of the problem is represented as

$$u = u_0 r, \quad \rho = \rho_0 r^{2(\omega-1)}, \quad T = T_0 r^2, \quad w = w(t, r, z). \tag{5.1}$$

Substituting representation (5.1) into system (1.1), we obtain

$$w_{rz} - 4\omega(w_z - u_0)/r - 3T_0^{-\omega}\rho_0(2\omega T_0 + u_0^2)/r = 0; \tag{5.2}$$

$$w_{rr} + 4w_{zz}/3 - T_0^{-\omega}\rho_0(w_t + ww_z)/r^2 + (2\omega + 1 - T_0^{-\omega}\rho_0 u_0)w_r/r = 0; \tag{5.3}$$

$$w_z + 2\omega u_0 = 0; \tag{5.4}$$

$$w_r^2 + 4w_z^2/3 - (T_0^{1-\omega}\rho_0 + 8u_0/3)w_z - 2c_V T_0^{1-\omega}u_0 r^{2(1-\omega)}$$
$$+ 4æ_0 T_0(\omega + 1) + 4u_0^2/3 - 2T_0^{1-\omega}\rho_0 u_0 = 0. \tag{5.5}$$

Integrating (5.4), we find

$$w = -2\omega u_0 z + f(t, r). \tag{5.6}$$

Substituting (5.6) into system (5.2)–(5.5) and analyzing the relationships obtained, we obtain $\omega = 0$ and $u_0 = 0$. In this case, Eq. (5.5) has the form $f_r^2 + 4æ_0 T_0 = 0$, wherefrom there follow $æ_0 = 0$ and $f_r = 0$ by virtue of conditions (1.3). Equation (5.3) yields $f_t = 0$, wherefrom we have $w \equiv w_0$. It is shown that this solution is reduced to invariant solutions constructed on the subalgebra with the basis X_1, X_3, and $X_5 - w_0 X_2$. Therefore, there are no nonreduced solutions for subalgebra 4.4.

Thus, all partially invariant solutions of rank 0 and defect 1 for a system describing axisymmetric motions of a viscous heat-conducting perfect gas are constructed in the paper.

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